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On the Mechanics of Growing Shells and Plates

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Recently, the interest in growing solids and modeling of growth has increased dramatically owing to the development of new technologies of manufacturing of new promising materials as well as to the extension of continuum mechanics approaches to nonclassical problems. Examples of growing bodies include growth of bones and tissues, solids with phase transformations, and growth of crystals and thin films. Despite various and in general distinct growth mechanisms, these problems have many common features. These include the *a priori* unknown position of the interface, ageing or changing of mechanical properties with time, possible absence of natural reference placement, nonlinearity related to the motion of interface, etc. Various approaches to the modeling of growth of solids can be found for example, in [1–9]. In particular, one has to distinguish between volumetric and interfacial types of growth. In what follows, we consider interfacial growth, where a solid grows by attaching particles or thin layers to its surface.

The aim of the lecture is to consider models of growing plates. This problem is important at least for two reasons. First, the mechanics of "growing" thin-walled structures is useful for modeling thin films growth in various manufacturing methods. The second reason is that 2D models do not require detailed information on the microstructure of a thin film across its thickness. Indeed, the effective mechanical properties of plates and shells inherit the microstructure of material as integral characteristics. Our approach is based on the application of two Euler laws for modeling thin-walled structures. First, we consider the balance of momentum and moment of momentum for a system of mass points and similar balance equations for a solid body. The momentum and the moment of momentum of n mass-points with respect to the point *o* with the radius-vector r_0 are B

$$\mathfrak{B}_m = \sum_{i=1}^{n} m_i \mathbf{v}_i \quad \text{and} \quad \mathfrak{M}_m = \sum_{i=1}^{n} (\mathbf{r}_i - \mathbf{r}_0) \times m_i \mathbf{v}_i$$

respectively. So we formulate two balances as follows

$$\frac{D}{Dt}\mathfrak{B}_b = \mathfrak{F}, \quad \mathfrak{F} \overset{\Delta}{=} \sum_{i=1}^{n} f_i; \quad \frac{D}{Dt}\mathfrak{M}_m = \mathfrak{C}, \quad \mathfrak{F} \overset{\Delta}{=} \sum_{i=1}^{n} (\mathbf{r}_i - \mathbf{r}_0) f_i. \qquad (1)$$

Here $\mathfrak{F}$ and $\mathfrak{C}$ are the total (resultant) force vector and the total torque (resultant moment) with respect to point *o*, respectively, $\mathbf{r}_i$ is the position of *i*th mass-point, $\mathbf{v}_i$ is its velocity. For any portion of a solid similar balance laws are valid

$$\frac{D}{Dt}\mathfrak{B} = \mathfrak{F}, \quad \mathfrak{F} \overset{\Delta}{=} \iiint_{v_p} \rho \mathbf{f}\, dv + \iint_{\partial v_p} \mathbf{t}\, da;$$
$$\frac{D}{Dt}\mathfrak{M}_m = \mathfrak{C}, \quad \mathfrak{F} \overset{\Delta}{=} \iiint_{v_p} \rho (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{f}\, dv + \iint_{\partial v_p} (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{t}\, da. \qquad (2)$$

Here **f** and **t** are the densities of the mass and surface forces forces, respectively.

We modell grows as a deposition of mass-points to the solid surface. In a similar way we formulate two-dimensional Euler's laws for nonlinear shells. In the case of a shell with base surface w we have

$$\frac{D}{Dt}\mathfrak{B} = \mathfrak{F}, \quad \mathfrak{F} \overset{\Delta}{=} \iiint_{\omega_p} \rho_s \mathbf{q}\, da + \iint_{\partial \omega_p} \mathbf{t}_s\, ds;$$
$$\frac{D}{Dt}\mathfrak{M}_m = \mathfrak{C}, \quad \mathfrak{F} \overset{\Delta}{=} \iiint_{\omega_p} \rho_s [(\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{q} + \mathbf{m}]\, da + \iint_{\partial \omega_p} [(\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{t}_s + \mathbf{m}_s]\, ds. \qquad (3)$$

Here **q** and **m** are the surface densities of the forces and the moments acting on the shell surface, and $\mathbf{t}_s$ and $\mathbf{m}_s$ are contour loads, see [10] for details.

The main idea of reduction procedure from 3D equations to 2D ones is similar to the widely use through-the-thickness procedure in the nonlinear theory of shells [11]. Here this reduction also includes the properties of mass points. We consider the Euler laws for the "solid—system of mass points" system in an arbitrary cylindrical region and for its 2D analogue. A comparison of the integral 3D and 2D formulations leads to relations between the 3D constitutive equations of this system and the 2D analogs. Various modes of mass point deposition are considered. Similarities between the models of growing shells and shells with surface stresses [12] are discussed.

References

[1] *N. Kh. Arutyunyan, A. V Manzhirov, and V. E. Naumov*. Contact Problems in Mechanics of Growing Solids [in Russian]. Nauka Publ., Moscow, 1991.

[2] *L. A. Taber*. Biomechanics of growth, remodeling, and morphogenesis. Applied Mechanics Reviews, 1995. Vol. 48, No. 8, pp. 487–545.

[3] *M. Epstein and G. A. Maugin*. Thermomechanics of volumetric growth in uniform bodies. International Journal of Plasticity, 2000. Vol. 16, No. 7, pp. 951–978.

[4] *A. DiCarlo and S. Quiligotti*. Growth and balance. Mechanics Research Communications, 2002. Vol. 29, No. 6, pp. 449–456.

[5] *M. B. Amar and A. Goriely*. Growth and instability in elastic tissues. Journal of the Mechanics and Physics of Solids, 2005. Vol. 53, No. 10, pp. 2284–2319.

[6] *A. Goriely and M. B. Amar*. Differential growth and instability in elastic shells. Physical Review Letters, 2005. Vol. 94, No. 19, pp. 198103.

[7] *K. Y. Volokh*. Stresses in growing soft tissues. Acta Biomaterialia, 2006. Vol. 2, No. 5, pp. 493–504.

[8] *A. Goriely, M. Robertson-Tessi, M. Tabor, and R. Vandiver*. Elastic growth models. In: Mathematical modelling of biosystems, 2008. pp. 1–44. Springer, Berlin, Heidelberg.

[9] *A. V. Manzhirov and S. A. Lychev*. The mathematical theory of growing solids: Finite deformations. Doklady Physics, 2012. Vol. 57, No. 4, pp. 160–163.

[10] *V. A. Eremeyev, L. P. Lebedev, and H. Altenbach*. Foundations of Micropolar Mechanics. Springer, Heidelberg, 2013.

[11] A. Libai, J. G. Simmonds. The Nonlinear Theory of Elastic Shells, 2nd edition. Cambridge University Press, Cambridge, 1998.

[12] *H. Altenbach and, V. A. Eremeyev*. On the shell theory on the nanoscale with surface stresses. International Journal of Engineering Science. 2011. Vol. 49, No. 12, pp. 1294–1301.