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Some Aspects about Moving Sharp Interfaces

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Introduction: Interface phenomena are of great importance in many areas (e.g., physics, mechanics, chemistry, and biology) and have been intensively studied over many years; e.g., see [1– 6]. A typical example of an interface dividing phases is the boundary of a melting ice piece surrounded by water (Stefan problem, see [7, 8]). Generally, interfaces can be classified (at least) by their geometry (sharp or diffusive), by their relation to the remaining body (consisting of the same body point for all time or not) as well as by their functionality (equipped with own thermodynamic behavior or not). Although many investigations have been performed, there remain open questions both in modeling as well as in investigation of the arising mathematical problems, which are as a rule of free-boundary type.

Aims of the contribution: We deal with the general (mathematical) modeling of sharp interfaces (i.e., two-dimensional surfaces within a three-dimensional body) without and with own thermodynamical activity (i.e. with interfacial densities, fluxes, and supplies). Questions of regularity of the arising quantities will be also addressed. Moreover, we present a classification of sharp interfaces taking their relation to the remaining body as well as their thermodynamic behavior into account. Finally, we discuss a model of a shrinking body (e.g., due to mechanical treatment) in the framework of sharp singular interfaces.

Some aspects of modeling of sharp interfaces: A three-dimensional material body is identified with its reference configuration $\overline{\Omega}$ (= closure of a bounded Lipschitz domain Ω representing the inner body points). At the beginning $t = 0$, Ω is assumed to be a disjoint composition of two sub-domains $\Omega_A(0)$,

$\Omega_B(0)$ representing the phases A and B and of an interface $\Gamma(0)$ separating $\Omega_A(0)$ and $\Omega_B(0)$:

$$\Omega = \Omega_A(0)\cup\Gamma(0)\cup\Omega_B(0), \Omega_A(0)\cap\Omega_B(0)=\varnothing, \Omega_A(0)\cap\Gamma(0)=\varnothing, \Omega_B(0)\cap\Gamma(0)=\varnothing.$$

Due to possible phase changes, there are “time-dependent changes in the reference configuration” ([4], [Chapter 4]). We assume that for all $t\in[0,T]$ or $t\in[0,\infty[$ ($T>0$ - given process time)

$\Gamma(t)=\bar{\Omega}_A(t)\cap\bar{\Omega}_B(t)\subset\Omega$ homeomorphic image of the unit sphere in $\mathbf{R}^3$ (2)

$$\Omega_A(t)\cap\Omega_B(t)=\varnothing, \Omega_A(t)\cup\Gamma(t)\cup\Omega_B(t)=\Omega.$$

The sets $\Omega_A(t)$ and $\Omega_B(t)$ represent those body points which at the instant t belong to the phase A and B, respectively, within the reference configuration (see Fig. 1, left and center).

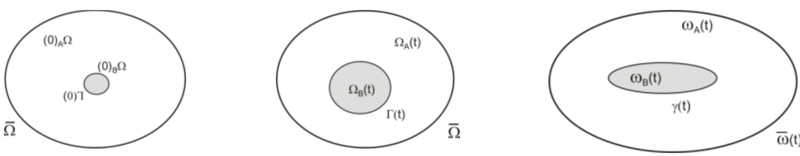

Figure 1: Cross-section of a two-phase body $\bar{\Omega}$ with time-dependent interface $\Gamma(t)$ in material representation at the beginning (left), at instant $t\in]0,T[$ (center), and in current representation at instant t. Note that generally the interface is not smooth.

Here the body motion χ $:\bar{\Omega}\times[0,T]\rightarrow\mathsf{R}^3$ is assumed to be continuous. It bijectively maps the reference configuration $\bar{\Omega}$ onto the current configuration $\bar{\omega}(t)$ for all $t\in[0,T]$ (see Fig. 1, right). As a consequence, for all $t>0$ the motion χ maps the (possibly smooth) interface $\Gamma(t)$ (within the reference configuration) onto its counterpart $\gamma(t)=(\Gamma(t),t)$. An interface is called *material* if it is formed by the same body points for all time:

$$\forall t > 0 \Gamma(t) = \Gamma(0), \gamma(t) = \chi(\Gamma(0), t).$$

Contrary to this, we call an interface *singular* if the relation (3) does not hold. Thus, a singular interface cannot be described by the body motion alone.

A crucial feature of modeling is that the motion χ is only smooth up to the interface from either side but generally *not across* it. As a consequence, the deformation gradient F and the body velocity υ are generally not continuous across the interface. Moreover, the local balances (mass, impulse, ...) at the interface contain jumps of bulk quantities. For interfaces with own thermodynamic behavior, these interfacial local balances are partial differential equations involving jumps (see [2, 4, 5]). At interfaces without own thermodynamic activity, there are only pure jump conditions (see [1, 3, 6]). Moreover, we deal with an intermediate type of interfaces bearing only supplies (type II in our denomination). This leads to nonhomogeneous jump conditions allowing a not too complex modeling.

A model of a shrinking body: As an example we present a model of a body undergoing a loss of material in the framework presented here. A workpiece may lose material due to mechanical treatment (turning, milling, e.g.). To avoid too high complexity, this process can be modeled as a two-phase body with a singular interface. One phase represents the shrinking body, and the other one its lost material. Assuming that this removed material has no essential retroactivity to the body, one can avoid complex modeling of the behavior of this second phase. Thus, in exchange, we consider a modified model, making the following assumptions. The part of the body's surface at which material is lost is regarded as a singular interface. Its velocity is caused by mechanical (and/or thermal) effects. The engineering details are extraneous here. At the interface, the loss of material is modeled by an interfacial mass supply (more precisely, by a sink). In view of this, there arise an interfacial loss of impulse, energy and entropy. All densities and supplies associated with the outside (i.e., the phase of removed material) are assumed to vanish. There may be normal fluxes from the outside at the interface.

The interface is considered as a type S-II interface, i.e., it is singular and only interfacial supplies are allowed which are in balance with the corresponding jumps of bulk quantities across the interface. The (shrinking) body may behave as a deformable solid (e.g., an elastoplastic material). As a result, we obtain a consistent model described by a coupled system of bulk equations with boundary conditions related to momentum and heat transfer at the singular surface.

Outlook: The final aim is to develop a model for singular interfaces with full thermodynamic activity and apply it to phase transformations in steel in the framework of mathematical homogenization.

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