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Growing Shells. Theoretical Modeling and Experimental Identification

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The technology of additive manufacturing are now under intensive development. The advantages of such technology are well known, however, their implementation causes a number of challenges. Among others, one can point to the problem for minimizing of the distortion of geometrical shape that is most significant with manufacturing of thin-walled structures. Present communication deals with the issues of mathematical modeling for the distortion of thin-walled solids caused by additive technological process and identification of parametric models from experimental data.

As it was shown in [1-2] the growing bodies can be viewed as a special class of inhomogeneous bodies whose inhomogeneity is caused by junction of incompatible deformed parts. In what follows, we use the concept of a body as a smooth manifold equipped with a *material connection* [3-4]. The notion of *material connection* formalizes the idea of a local uniform reference configuration that brings an infinitesimal neighborhood of a material point into some uniform, typically natural strain state. In the simplest cases, one can bring the entire body into a uniform state by some global configuration. In general, one cannot simultaneously bring infinitesimal neighborhoods of all material points into a uniform state by a smooth mapping. For these a materially uniform body, i.e. a body all of whose material points are of the same kind, possesses some intrinsic (structural) inhomogeneity.

Consider growing bodies representable as continuous families of nested bodies [4]. A *layerwise growing body* is a continuous family of bodies monotone with respect to inclusion, i.e.

$$\mathfrak{C} = \{\mathfrak{B}_\alpha\}_{\alpha\in I}, \quad \mathfrak{B}_\alpha \subset \mathfrak{B}_\beta \quad \text{for } \alpha < \beta, \tag{1}$$

where $I = (a,b) \subset \mathbb{R}$ is an open interval. Bodies represent themselves in the physical space $\mathcal{E}$ as *shapes*. Every shape is the image of a *configuration* $\kappa_\alpha : \mathfrak{B}_\alpha \to \mathcal{B}_\alpha$. To each element of the family (1), we assign two shapes, a reference shape $\mathcal{B}_\alpha^R = \kappa_\alpha^R \mathfrak{B}_\alpha$ and an actual shape $\mathcal{B}_\alpha = \kappa_\alpha \mathfrak{B}_\alpha$.

We assume that the bodies $\mathfrak{B}_\alpha$ are materially uniform, simple, and elastic, so that their response can be described by the constitutive equation $\boldsymbol{\sigma}_\alpha = \mathfrak{S}(\mathbf{H}_\alpha)$, where $\mathfrak{S}(\dots)$ is the response functional. The smooth tensor field $\mathbf{H}_\alpha$ represents the local distortion and admits the multiplicative decomposition $\mathbf{H}_\alpha = \mathbf{F}_\alpha \circ \mathbf{K}_\alpha$, where $\mathbf{F}_\alpha$ is the conventional strain gradient, i.e., the linearization of the mapping $\gamma_\alpha : \mathcal{B}_\alpha^R \to \mathcal{B}_\alpha$. The tensor field $\mathbf{F}_\alpha$ is compatible: there exists a vector field with gradient $\mathbf{F}_\alpha$. This property is not true in general for the smooth tensor field $\mathbf{K}_\alpha$, which is called an *implant field* [4] and is a field of linear transformations that combines the incompatible infinitesimal parts without gaps.

Let $\mathbf{G}_\alpha$ be the family of vector fields on $\mathfrak{B}_\alpha$ specifying the external bulk forces. Although the form of momentum equations in the bulk is identical to the classical one

$$\nabla \cdot \boldsymbol{\sigma}_\alpha + \mathbf{G}_\alpha - \rho_\alpha \partial_t \mathbf{v}_\alpha = \mathbf{0}, \tag{2}$$

the equations on the boundary $\partial\mathcal{B}_\alpha$ are distinctive:

$$\mathbf{n}_\alpha \cdot \boldsymbol{\sigma}_\alpha|_{\partial\mathcal{B}_\alpha} = \mathbf{p}_\alpha + V[\rho_\alpha \mathbf{v}_\alpha], \quad \mathbf{A}_\alpha \cdot \boldsymbol{\sigma}_\alpha \cdot \mathbf{A}_\alpha|_{\partial\mathcal{B}_\alpha} = \mathcal{T}_\alpha. \tag{3}$$

Here $\mathbf{v}_\alpha$ is the velocity field, ρ_α is the mass density, V is the growth velocity, and $\mathbf{p}_\alpha$ and $\mathcal{T}_\alpha$ are predefined families of fields. The former defines the traction on the boundary $\partial\mathcal{B}_\alpha$, and the latter defines the *tension* of the boundary treated as an elastic material surface in contact with a three-dimensional elastic solid [5]. The vector $\mathbf{n}_\alpha$ is the unit outward normal to $\partial\mathcal{B}_\alpha$, $\mathbf{A}_\alpha$ is the projection onto the corresponding tangent plane, and $[\dots]$ is the jump of a field on the surface $\partial\mathcal{B}_\alpha$.

In the present communication we assume that all bodies $\mathcal{B}_\alpha^R$ belong to the class of *transversely growing shell* [6], i.e. each reference shape $\mathcal{B}_\alpha$ is bounded by an overall ruled surface S_0 and pairs of face surfaces S_α^+ and S_α^- whose shape and position depend on the evolution parameter α.

In the framework of any shell theory, one has to choose certain representation of the displacements by an ordered set of n kinematic fields $\{\boldsymbol{\xi}_\alpha^k\}_{k=1}^n$ defined on a 2D manifold S (the reduction surface) and introduce integration over thickness, i.e., along the normal coordinate z from h_α^- to h_α^+. This procedure gives the *stress resultant tensor* $\mathcal{T}_\alpha$ and the *stress couple tensor* $\mathcal{M}_\alpha$:

$$\mathcal{T}_\alpha = \int_{-h_\alpha^-}^{h_\alpha^+} \hat{\boldsymbol{\sigma}}_\alpha \, dz, \quad -\mathcal{M}_\alpha = \int_{-h_\alpha^-}^{h_\alpha^+} \hat{\boldsymbol{\sigma}}_\alpha \times z\mathbf{n} \, dz, \quad \hat{\boldsymbol{\sigma}}_\alpha = \hat{\mathfrak{S}}(\boldsymbol{\xi}_\alpha^1, \dots, \boldsymbol{\xi}_\alpha^n; \mathbf{K}_\alpha; z). \tag{4}$$

Here the $\hat{\boldsymbol{\sigma}}_\alpha$ are some "reduced" stresses related to the stress tensor components by a reduction procedure that depends on the chosen theory [8]. In this regard, one has the following mappings:

$$\mathcal{T}_\alpha = \hat{\mathcal{T}}(\boldsymbol{\xi}_\alpha^1, \dots, \boldsymbol{\xi}_\alpha^n; \mathbf{K}_\alpha; h_\alpha^-, h_\alpha^+),$$
$$\mathcal{M}_\alpha = \hat{\mathcal{M}}(\boldsymbol{\xi}_\alpha^1, \dots, \boldsymbol{\xi}_\alpha^n; \mathbf{K}_\alpha; h_\alpha^-, h_\alpha^+). \tag{5}$$

According to common considerations typical of shell theories, the tensor fields (4) should satisfy the equations of motion

$$\nabla_s \cdot \mathcal{T}_\alpha + \mathbf{g}_\alpha - \boldsymbol{\Xi}_1 \cdot \partial_t \boldsymbol{\xi}_\alpha - \boldsymbol{\Theta}_1 \cdot \partial_t^2 \boldsymbol{\xi}_\alpha = \mathbf{0},$$
$$\nabla_s \cdot \mathcal{M}_\alpha + (\mathcal{T}_\alpha)_\times + \mathbf{m}_\alpha - \boldsymbol{\Xi}_2 \cdot \partial_t \boldsymbol{\xi}_\alpha - \boldsymbol{\Theta}_2 \cdot \partial_t^2 \boldsymbol{\xi}_\alpha = \mathbf{0}.$$

Here ∇_s is an appropriate surface Hamilton operator, $(\dots)_\times$ is a vector invariant of a tensor, and $\mathbf{g}_\alpha$ and $\mathbf{m}_\alpha$ are external force and moment fields distributed over the reduction surface. The matrices $\boldsymbol{\Theta}_1$ and $\boldsymbol{\Theta}_2$ are the inertia matrices, and $\boldsymbol{\Xi}_1$ and $\boldsymbol{\Xi}_2$ represent "dissipative" terms due to the last expression in Eq. (3).

In very simple cases, the implant field $\mathbf{K}_\alpha$ is predefined for all values of the evolution parameter α. But in most cases we have to determine $\mathbf{K}_\alpha$ from some conditions corresponding to the evolution process, particularly from predefined displacements of the face boundaries or their membrane tension:

$$\mathbf{A}_\alpha \cdot \hat{\boldsymbol{\sigma}}_\alpha \cdot \mathbf{A}_\alpha|_{\Gamma_\alpha^+} = \boldsymbol{\tau}_\alpha^+, \quad \mathbf{A}_\alpha \cdot \hat{\boldsymbol{\sigma}}_\alpha \cdot \mathbf{A}_\alpha|_{\Gamma_\alpha^-} = \boldsymbol{\tau}_\alpha^-. \tag{6}$$

Note that $\boldsymbol{\tau}_\alpha^+$ and $\boldsymbol{\tau}_\alpha^-$ actually are the surface stresses that act on the face boundaries S_α^+ and S_α^-. Thus, the equations (6) turn the evolution problem into a family of problems for shells with surface tension [9].

The distortion of thin-walled structure depends on implant field $\mathbf{K}_\alpha$. In turn, $\mathbf{K}_\alpha$ is implemented by the additive process. In this regard one could be tasked with experimental identification of the distortion in the various conditions of an additive process in the framework of proposed model. Experimental procedure is based on holographic interferometry of shape distortion during the stereolithographical process. The experimental setup includes an open stereolithographical system implemented on a vibration-isolated table together with the installation for holographic interferometry. The form of the object being created is designed to be a thin-walled cylindrical tube with base. The interferometrical system records the evolution of displacements field for the base in time. Measured data and the corresponding inverse problem solution allow us to identify the field $\mathbf{K}_\alpha$ with respect to various regimes of additive process.

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